Inchworm and a Half:

Developing Fraction and Measurement Concepts Using Mathematical Representations

Children's conceptual understanding, strategic competence, and procedural fluency in mathematics are highly influenced by experiences in the early grades. During these years, many important mathematics concepts, including measurement and rational-number sense, should be a part of children's informal investigations. By informally exploring the same concepts in a variety of ways—with concrete manipulatives, visual models, and abstract representations—first graders can connect their initial youthful notions to sophisticated mathematical thinking.

That children understand processes of measurement using nonstandard units and select appropriate units for objects they are measuring is a broad goal of NCTM's Measurement Standard for Grades Pre-K–2 (NCTM 2000). The Number and Operations Standard states that young children should understand ways of representing numbers and relationships among numbers. Specifically, they should "understand and represent commonly used fractions, such as 1/4, 1/3, and 1/2" (NCTM 2000, p. 78).

By Patricia S. Moyer and Elizabeth Mailley

Patricia S. Moyer, pmoyer@gmu.edu, teaches mathematics education courses in the Graduate School of Education at George Mason University in Fairfax, Virginia. She is the director of the Mathematics Education Center, where her research interests focus on uses of representations in school mathematics and mathematics teacher development. Elizabeth Mailley, bnbmailley@aol.com, teaches first grade at Westlawn Elementary School in Falls Church, Virginia. She enjoys teaching lessons that link mathematics and literature and is interested in using representations to make challenging mathematics topics attainable for first graders.

This article discusses the use of the book *Inchworm and a Half* (Pinczes 2001) to stimulate children's investigations in measurement and number. During a two-day lesson based on the book, children used physical models and visual representations to build on their intuitive understandings of measurement units and develop an awareness of equivalence. This article describes the planning decisions that were part of developing the lesson, the introduction of mathematics concepts in the text during the lesson, the development of measurement skills using vegetables depicted in the book, and the exploration of equivalent relationships using paper worms. We photographed children and their work during the two-day lesson, and audiotape recorders captured their conversations during their work and sharing time. The following sections describe the lesson, including children's work samples and conversations.

About the Book

Inchworm and a Half by Elinor Pinczes (2001) is the story of a worm who enjoys measuring vegeta-

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bles in a garden. She measures hot peppers, pole beans, and eggplants, singing, "Squirmy, wormy, hoopity-hoop! I measure everything, loopity loop" (p. 4). The worm measures happily until one day when the unthinkable happens: She discovers that she cannot accurately measure a cucumber that is two worm lengths and a little bit more. Luckily a worm half her size appears, and together they are able to measure the cucumber.

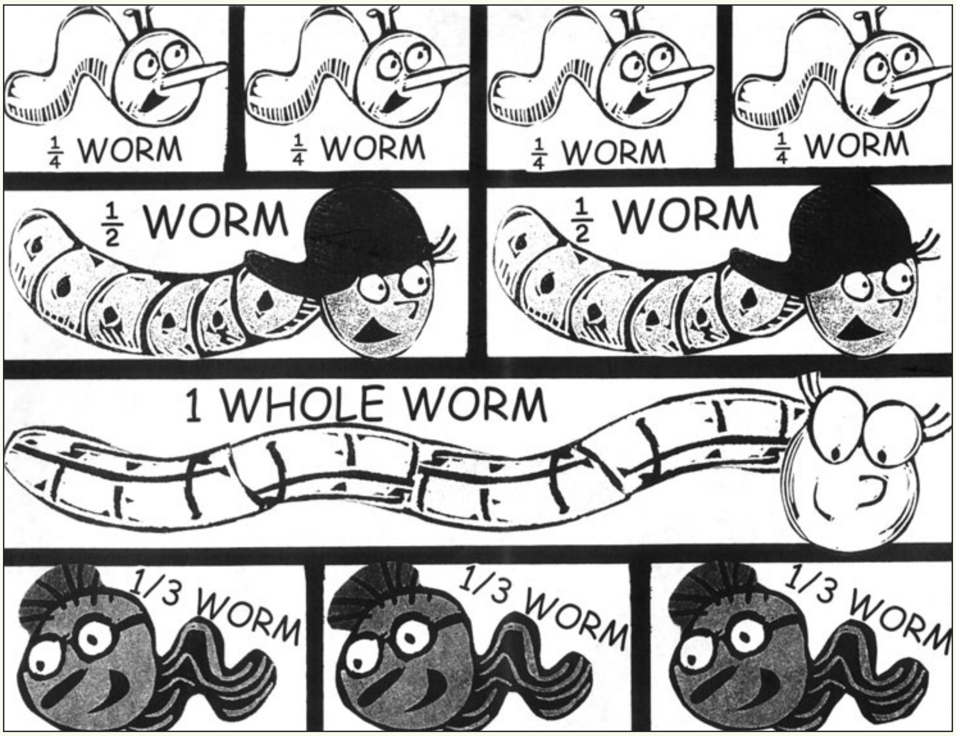
As the book progresses, worms one-third her size and one-fourth her size join in the measuring, and together the four worms (1 whole worm, 1/2 worm, 1/3 worm, and 1/4 worm) are able to measure all the vegetables in the garden. The text is written in a lyrical style, and the illustrations by Randall Enos, particularly the facial expressions of the worms, make the book inviting and enjoyable for young children who can predict what will happen next in the story.

Planning the Lesson

Many resources exist for teachers to use in planning a mathematics lesson that uses a children's book (Burns 1992; Griffiths and Clyne 1988). During several brainstorming sessions, we discussed possible mathematics topics and appropriate extensions that could be taught with the text. We noted that in addition to measuring vegetables, the worms in the story also represent different fractional amounts in relationship to one another. Because the text combines both measurement and fraction concepts, we decided to address each concept in a two-part lesson. Day one would focus on arbitrary or nonstandard-length measurement and day two would focus on equivalent fractional relationships. Prior to the introduction of operations (addition, subtraction, multiplication, and division) with fractions in subsequent grades, children in first grade explore fraction concepts such as equivalent fractions in the context of measurement that involves nonstandard or standard units. In previous lessons, the children used several different nonstandard length measurements. They explored nonstandard measuring devices such as paper clips, chain links, and Unifix cubes as referents for measuring classroom objects. These experiences taught them important concepts about aligning the measuring device with the object they were measuring. This group of first graders also had previous experience with concrete and pictorial representations

Figure 1

The paper cut-outs the children used



and set and region models using fractions. They had some general knowledge about fractional quantities and were familiar with how to write a fraction. Representations that connect abstract ideas with visual and physical models help bring these challenging concepts within a first grader's reach.

Our ideas for day one came directly from the content of the book: The worms in the text were measuring vegetables, so we decided to introduce the measurement concepts by having the children measure vegetables as well. To prepare for the first part of the lesson, we gathered vegetables, created a vegetable recording sheet, and designed paper worms based on the worms in the book. The children would measure real vegetables using paper cut-outs that represented, both in appearance and relational size, the cartoon-character worms from the story (see **fig. 1**). These paper worms could be used individually or together to measure the lengths of different vegetables.

Children's early experiences with fractions often focus on region models because they are easier for children to learn. Unfortunately, some teachers do not move beyond this basic model for fractions, and they overlook other models. For example, introducing the *measure* interpretation of fractions (Kieren 1980) is one way to help children understand concepts involving the density of the rational numbers. Children understand the density of rational numbers when they are able to perform partitions other than halving, identify fractions between two given fractions, and use given unit intervals to measure any distance from the origin (Lamon 1999). Therefore, we chose to develop and use fractional models for length measurement by representing the different lengths of the worms in the story: 1 whole worm, 1/2 worm, 1/3 worm, and 1/4 worm.

In addition to exploring the concept of length measurement, the book's dialogue compares the fractional relationships of the worms' sizes. For example, the half-inch worm compares himself to the one-whole worm when he says, "At just half your size, I'm a one-half-inch fraction, you see." The illustration verifies this comment, helping the reader both see and verbalize the relationship between the two worms. We wanted children to focus on these comparisons between the lengths of the worms while we also introduced ideas of fractional relationships. Therefore, we designed day two's lesson so that children could manipulate and compare the paper-worm cut-outs to find sets of equivalent values, then use abstract representations to record these findings on construction paper.

Exploring Mathematical Concepts in the Text

On both days, the teacher began the mathematics activities by reading the story to the children. As she read the text, she asked questions that encouraged children to examine the mathematics in the story. Her questions prompted the children to examine the relationships among the worms and make comparisons. For example, when the text showed the whole worm compared to the one-half worm, the teacher asked, "What do you notice about these two worms?" and "How many of these [worms] would you need to make one of those?" She asked questions that helped the children think about the lengths of the vegetables, such as "Can anyone tell how long this leaf is that she measured by looking at how many loops she made?" and "Why do you think they couldn't finish measuring the carrot?" The teacher also asked questions that helped children look for patterns. For example, after the one-half and one-third worms appeared, the teacher asked the children to predict the fractional measurement of the next worm.

Reading the book prompted interesting questions and conversations from the children. At one point in the story, when one of the worms said, "My length's one-third-inch on the dot" (p. 19), a child asked, "How come he's a dot?" The teacher used this opportunity to develop the children's vocabulary by addressing concepts and expressions of precision (such as *on the dot*) that relay ideas about measurement. Other children made observations such as "On the cover it has different kinds of names, but it should be called *The Fraction Worms*" and "They're making a worm chain." Another child wanted to know, "What's an asparagus spear?" These comments demonstrated that the children were making observations, predictions, and generalizations from the text and illustrations that included and went beyond measurement and fractional concepts. The exchanges between the teacher and the children built on their curiosity and were an important bridge to the next part of the lesson.

Figure 2

Vegetable worksheet for recording measurements

Name ______________________

Draw a picture of the vegetable and record your measurement on the line.

1. celery ______
2. carrot ______
3. asparagus ______
4. bean ______
5. cucumber ______
6. zucchini ______

Using Paper Worms as Nonstandard Length Models

After the teacher finished reading the story on the first day, she showed the students examples of some of the different vegetables in the book and named each one. She told the children that they would be measuring vegetables just as the worms in the story had done. The teacher showed the children paper cut-outs of 1 whole worm, 1/2 worm, 1/3 worm, and 1/4 worm. The children were familiar with these symbolic representations from previous lessons, and the teacher's comparison of the paper worm strips reinforced their understanding of the fractional relationships among the worm lengths. For example, she asked the children to compare the length of two one-half worms to the one-whole worm and to find other combinations of worms that might make a one-whole worm. She also asked the children to compare which worms were larger or smaller and displayed the different worms as examples. She was careful to point out to the children that their paper worms were not called "inchworms" because they were much bigger than one inch.

While the children sat with the teacher in a circle on the floor, the teacher modeled how to measure a stalk of celery using the paper worms. She demonstrated how to measure the celery by lining up the end of the paper worm with the end of the stalk of celery. One child suggested that the teacher start with the one-whole worm to begin the measurement because, she explained, "It's almost as big as the celery." The teacher then asked, "Can this worm measure the celery all by itself?" The children replied, "No." After the teacher and the children experimented by adding the one-fourth worm to the one-whole worm, they determined that the stalk of celery was about one and one-fourth worms long. The teacher measured a carrot as another example for the children, and she showed them how to record the measurements on the vegetable worksheet. She also explained that not all the vegetable measurements would be exact but the children should use the closest measurement that they could find.

When the children returned to their table groups, the teacher gave each of them a vegetable worksheet to record their measurements (see **fig. 2**). The following vegetables were in the center of each table group: a stalk of celery, a carrot, an asparagus spear, a green bean, a cucumber, and a zucchini. The children began to use the worms to measure the vegetables (see **fig. 3**). On their vegetable worksheets, they drew pictures of each vegetable next to the vegetable's name and recorded their measurements.

Prior to this lesson, we discussed how the children might record the worm lengths when they combined different fractional amounts to measure a vegetable. For example, how would they record a measurement when they used a one-half worm and a one-fourth worm to measure a carrot? We decided that we wanted to see what strategies the

Figure 3

Using the worms to measure the vegetables

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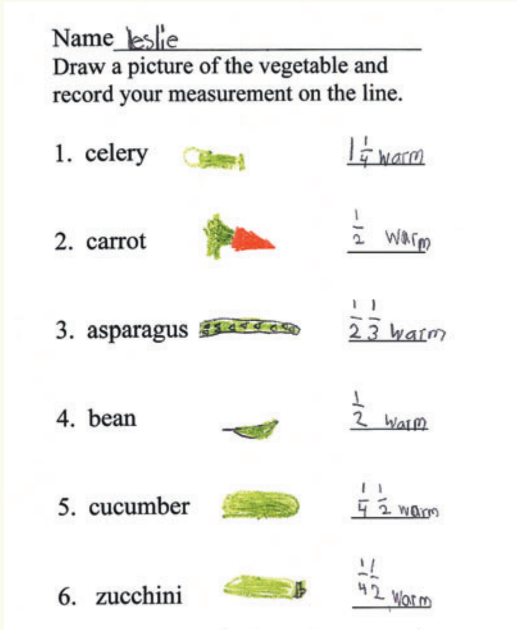
Figure 4

Many students recorded two fractional lengths side by side.



Name leslie

Draw a picture of the vegetable and record your measurement on the line.

1. celery 1 1/4 warm
2. carrot 1/2 warm
3. asparagus 1/2 1/3 warm
4. bean 1/2 warm
5. cucumber 1/4 1/2 warm
6. zucchini 1/4 1/2 warm

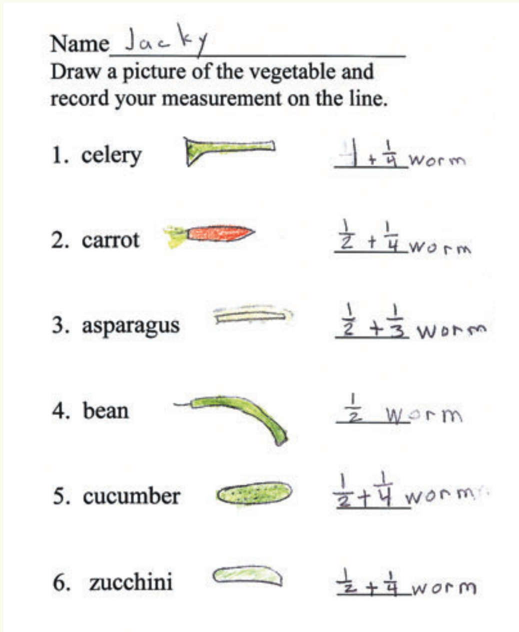
children would devise for recording these measurements, so we did not give them specific instructions on this part of the task. Although teacher-provided representations are important during instruction for learning the conventional aspects of representation (such as fractional symbols), allowing student-generated representations can enhance and deepen the meaning of fractions and other mathematics concepts for young children (Lamon 2001).

Throughout the lesson, the children were careful to measure from one end of the vegetable to the other, and they often tested worms of different lengths to determine which would give them the most accurate measurements. Most of the children recorded the fractional amounts using the word *worm* as a label for their numbers. When they used two fractional lengths together, such as 1/4 and 1/2 or 1/4 and 1/3, almost all of them wrote the two fractions side by side (see **fig. 4**). One student, however, used addition signs to join the two fractional amounts (see **fig. 5**). Another child combined 1/3 and 1/3 and wrote "2/3 worm" for the measurement of the carrot, and he also used the word *about* to show that the measurement was approximate (see **fig. 6**). The children appeared to develop meaning for these numbers through their own interactions with the length models and their interpretations of how to record the measurements.

After the children measured the vegetables, the teacher called them together as a group to share their measurements with one another. Because young children frequently possess much greater knowledge than they are able to express in writing, the sharing and discussion that follow their investigations are an important part of extending their learning. During their discussions on the first day, the students recognized that they were using different measurements for the same vegetables, so the teacher led a class discussion about reasons for these different results. The students conjectured that some vegetables might be different sizes at different table groups and that using paper worms was not a precise measurement, only an approximate measurement. They also talked about why some of the children's answers looked different when written as symbols, although they were actually the same length amounts. For example, the teacher compared one child's answer of 1/2 with another child's answer of 1/4 and 1/4 by using the one-half and one-fourth worms to compare the lengths. Continuing the comparison, the teacher asked the children if another way existed to write 1/4 and 1/4 so that they were combined. One of the children said, "You could have said it was two-fourths." The teacher summed up this discussion by saying, "Any of these are correct because the bean does measure one-half worm, the bean does measure one-fourth plus one-fourth, and we know that one-fourth and one-fourth is two-fourths so it measures two-fourths also." The teacher finished the lesson on the first day by recording examples of the children's measurements and showing them how they could put an addition sign between the fractional amounts to show that the two numbers were combined. These ideas led into the lesson on day two.

Figure 5

Jacky used addition signs to record the measurements.


Name Jacky

Draw a picture of the vegetable and record your measurement on the line.

1. celery 1 + 1/4 worm
2. carrot 1/2 + 1/4 worm
3. asparagus 1/2 + 1/3 worm
4. bean 1/2 worm
5. cucumber 1/2 + 1/4 worm
6. zucchini 1/2 + 1/4 worm

Exploring Equivalent Relationships

On the second day, the teacher read the story to the children again. After the measurement activities of the first day, her questions and the children's comments included new ideas about measurement concepts that the book presented. For example, because they had measured vegetables with the paper worms, the children now were able to verbalize measurement lengths more accurately. At the point in the book at which the inchworm and the one-half inchworm measure the beet together, the teacher pointed to the one-whole and one-half worms and asked, "If this beet is one of these worms and one of these worms long, how long is the whole beet?" The children responded, "One and a half." The teacher pointed out phrases in the book, such as "For every loop made by the inchworm, the shorter worm had to make two" (p. 16), and asked what the statements meant. This encouraged the children to look at the relationships between the lengths of the worms. In this example, discussion focused on the one-half inchworm making two loops for every one loop made by the inchworm.

The teacher revisited ideas from the previous day when children's measurements were written using different numbers, such as the "one-fourth plus one-fourth" or "one-half" or "two-fourths" measurement for the bean. She introduced the word *equivalent* to describe the equivalent values the children would be writing during day two and asked them to think of another word in mathematics that was similar to *equivalent*. Although the children were only in first grade, one perceptive child offered the word *congruent* as a similar word and another child suggested *equal*. Next, the teacher showed the students the one-whole paper worm and asked them to find another worm or worms that were equivalent to the one-whole

Figure 6

Nick combined fractions and noted approximations.

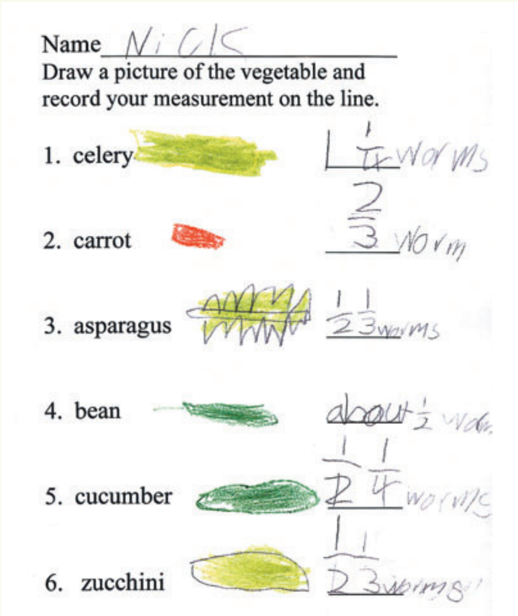
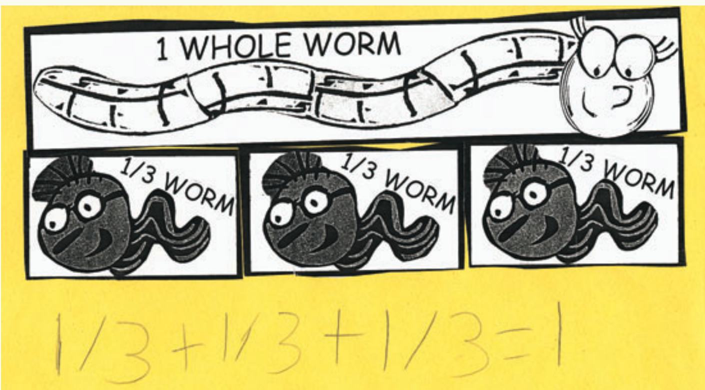


Figure 7

Finding equivalent sets



worm, noting, "I want them to be equal in length." One of the children suggested "two halves," so the teacher used two of the one-half worms, placing them end to end above the one-whole worm on her chart board to show that two of the one-half worms were as long as a one-whole worm. The teacher and the children discussed how they might record that the length of the one-whole worm and the length of two half worms was the same. The children realized that they could use addition and equals signs to show this relationship. They determined that 1/2 worm plus 1/2 worm equals 1 whole worm, so the teacher wrote 1/2 + 1/2 = 1 on the chart paper. The teacher told the children that when they returned to their desks, they would find three sets of the worms and a large piece of construction paper on each desk. Their task was to use the paper worms to find four different sets of equivalent lengths and then write the corresponding equation to represent the relationships they found.

When they found sets of worms that were equivalent, such as three one-third worms and a one-whole worm, they glued these worms end to end and placed one set above the other on their construction paper. Then they recorded their equations (in this case, 1/3 + 1/3 + 1/3 = 1) on the paper below the worms (see **fig. 7**). At one table, children were excited when they found an equivalent value: "Look! Those match! Look!" They even experimented with the new mathematical language that the teacher had taught them as one child repeated, "Equivalent! E . . . quivalent! E . . . quivalent!" As the following exchange shows, the children also were using problem-solving and estimation strategies:

Student 1. Try to do this one. This one is hard.
Student 2. But you know what? We can always use another one.
Student 1. Yeah, but that's gonna be too big.

Several children verbalized aloud as they wrote their equations on the construction paper. Examples of these verbalizations to others at the table or to the teacher included "One-third plus one-third plus one-third is one whole," "One-fourth plus one-fourth equals one-half," and "One-fourth plus one-fourth plus one-half equals one whole." Through these articulations, the students explored the names for equivalent values, and their written equations and the paper worms served as representations of their work.

Most of the children represented the following equations:

1/2 + 1/2 = 1
1/4 + 1/4 = 1/2
1/4 + 1/4 + 1/4 + 1/4 = 1
1/3 + 1/3 + 1/3 = 1

About one-third of the children represented the equation as 1/4 + 1/4 + 1/2 = 1 (see **fig. 8**). Other equations were more unique; only one or two children represented them. Examples of these included the following:

$$1/2 + 1/4 + 1/4 = 1/3 + 1/3 + 1/3$$
$$1/3 + 1/3 + 1/3 = 1/2 + 1/2$$
$$1/4 + 1/4 + 1/4 + 1/4 = 1/3 + 1/3 + 1/3$$
$$1/4 + 1/4 + 1/4 = 1/2 + 1/4$$

During this class session, none of the children made the common errors that students usually make when calculating the addition of fractions, such as adding both the numerators and denominators to get the incorrect equation of 1/2 + 1/4 = 2/6. Children did not make these typical errors because we did not ask them to add the fractional symbols; instead, we asked them to use the equals sign to show an equivalent relationship among the paper worms. The focus was on equivalence and relationships among the worms, rather than on performing a calculation. For first graders, the ability to express a fractional relationship in the form of an equation at this level of complexity would have been unlikely without the use of the paper worms as representations. Although these concepts will be developed more deeply in the upper elementary grades, this early experience gave meaning to the rational numbers that the students were using and served as an introduction to important concepts about equivalence for later mathematical development.

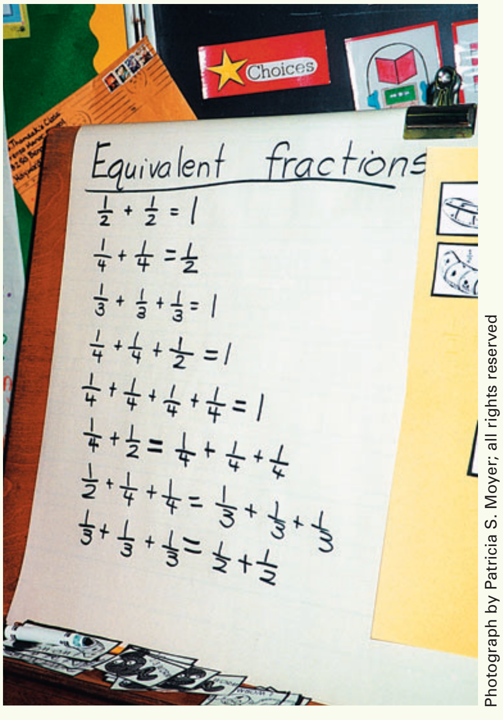
At the end of these investigations, the teacher called the children together as a group so they could share their equations with one another. The teacher recorded the children's number sentences on chart paper (see **fig. 9**). During their sharing, the children's comments showed the foundations of many important mathematical principles. For example, as one child observed the equations that the teacher recorded, she commented, "One-half equals one-fourth plus one-fourth, and one-fourth plus one-fourth equals one-half," revealing an important perception about equivalence. Another child's comment became a catalyst for discussing the concept of balancing equations and equivalence. She said to the teacher, "After she puts the equals sign, she shouldn't put any more. It should be her answer." This led to a discussion with the children about what the equals sign indicates in a number sentence and how the worms that the children compared were equal in length. During the discussion, the teacher held up student papers as visual models to explain this complex idea. They discussed that the equals sign is not a marker to show that an answer is coming but an indicator that both sides of the equation represent a value that is the same. This foundational understanding of equivalence is essential for students' later success in the study of algebra.

Figure 8

One student's representation of the equation

Figure 9

The students' number sentences



Reflections

Although most first-grade children cannot use symbolic representations alone to compare fractions with unlike denominators, using the measure interpretation of fractions (Kieren 1980) and its corresponding representations enabled these children to effectively make those comparisons. This method of interpreting fractions built on and extended principles of measurement with which the children were familiar. In this lesson, children learned how to choose a fractional unit that provided the most precise measurement, how to use several different fractional amounts to measure one object, and how to record fractions as units of measurement. Many other ways exist for teachers to foster the development of rational-number sense using the measure interpretation of fractions. That a unit of measure can always be divided up into smaller sub-units for a more precise measurement reading is an important concept to develop. Children can use number lines, rulers, or liquid containers and locate fractions between two numbers on the measuring instrument to develop this skill. Classroom activities—such as the successive partitioning of number lines, areas, and volumes—can connect these mathematical ideas in flexible ways and support the future study of the density of numbers and rational numbers as arbitrary divisions of a unit.

Fractions are *relational* numbers—that is, they express mathematical relationships between two discrete or continuous quantities—and are cognitively complex for students to learn. The paper worms supported the children as they generated new ideas about these concepts and laid their foundational understanding of equivalence. In this lesson, children learned how to use mathematical symbols to record an equivalent relationship, combined fractions to equal one whole, and used the equals sign to show an equivalent relationship rather than an answer. Frequent and varying opportunities for recognizing and creating equivalent representations for the same number help give children a broad and deep foundation for meanings and operations with rational numbers.

Fractional benchmarks such as 1/2, 1/3, and 1/4 are integral to children's development of rational-number sense. Understanding that different interpretations of fractions exist—from part-whole comparisons and measures to operators, quotients, and ratios and rates—further contributes to children's robust understanding of the rational numbers (Kieren 1980). Ultimately, the ability to compare fractions leads to greater facility with proportionality, or the ability to think flexibly about quantities and their relationships. Using a variety of representations in different mathematical situations supports the sense-making process and the development of these connections, furthering children's mathematical thinking and building their understanding of number relationships.

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